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HOMOGENEOUS CONVEX DOMAINS OF NEGATIVE SECTIONAL CURVATURE

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Let Ω be an affine homogeneous convex domain in a finite dimensional real vector space V, not containing any full straight line. Then we know that Ω admits an invariant volume element

$$v = K dx^1 \wedge \cdots \wedge dx^n$$

and that the *canonical bilinear form*

$$D\alpha = \sum_{i,j} \frac{\partial^2 \log K}{\partial x^i \partial x^j} dx^i dx^j$$

defines an invariant Riemannian metric on Ω, [2], [6]. In this note we prove the following theorem.

Theorem. *An affine homogeneous convex domain Ω not containing any full straight line has negative sectional curvature with respect to $D\alpha$ if and only if Ω is the interior of a paraboloid*:

$$y^0 - \frac{1}{2} \sum_{i=1}^{n-1} (y^i)^2 > -1 ,$$

where $\{y^0, y^1, \cdots, y^{n-1}\}$ is an affine coordinate system of V.

We first recall the construction of clans from homogeneous convex domains, [6]. In the following we assume that a homogeneous convex domain Ω contains the zero vector 0. Let G be a connected triangular affine Lie group which acts simply transitively on Ω, and let $\mathfrak{g}$ be the affine Lie algebra corresponding to G. For $X \in \mathfrak{g}$, we denote by $f(X)$, $q(X)$ the linear part and the translation vector of X respectively. Since q is a linear isomorphism of $\mathfrak{g}$ onto V, for each $x \in V$ there exists a unique $X_x \in \mathfrak{g}$ such that $q(X_x) = x$. We define an operation of multiplication in V by the formula

$$(1) \qquad x \cdot y = f(X_x)y \qquad \text{for } x, y \in V .$$

Then we have

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$$(2)\qquad [L_x, L_y] = L_{x\cdot y - y\cdot x}\ ,$$

where $L_x y = x\cdot y$, or equivalently

$$(2')\qquad x\cdot(y\cdot z) - (x\cdot y)\cdot z = y\cdot(x\cdot z) - (y\cdot x)\cdot z\ .$$

We put

$$(3)\qquad \alpha_0(x) = \mathrm{Tr}\, L_x\ ,$$

and identify the tangent space of Ω at 0 with V. Then the value of $D\alpha$ at 0 gives an inner product $\langle\ ,\ \rangle$ on V such that

$$(4)\qquad \langle x, y\rangle = \alpha_0(x\cdot y)\ .$$

By (2′) and (4) we get

$$(5)\qquad \langle x\cdot y, z\rangle + \langle y, x\cdot z\rangle = \langle y\cdot x, z\rangle + \langle x, y\cdot z\rangle\ .$$

The algebra V together with the linear function α_0 is said to be a *clan* corresponding to Ω. If we define a bracket operation in V by

$$(6)\qquad [x, y] = x\cdot y - y\cdot x\ ,$$

then V is a Lie algebra with respect to this bracket operation and q is a Lie algebra isomorphism of $\mathfrak{g}$ onto V. Therefore we may identify $\mathfrak{g}$ with V by means of q. Following Nomizu [4], we shall express the Riemannian connection, the curvature tensor and the sectional curvature of Ω in terms of its clan V; those expressions were originally obtained by Y. Matsushima (unpublished).

Proposition 1. *The Riemannian connection ∇ for $D\alpha$ is given by*

$$\nabla_x y = \tfrac{1}{2}(L_x - {}^tL_x)y\ ,$$

i.e., ∇_x is the skew symmetric part of L_x.

Proof. According to [4], we have

$$\nabla_x y = \tfrac{1}{2}[x, y] + U(x, y)\ ,$$

where $2\langle U(x, y), z\rangle = \langle [z, x], y\rangle + \langle x, [z, y]\rangle$. By (5), (6), we get

$$\begin{aligned}
2\langle U(x, y), z\rangle &= \langle z\cdot x - x\cdot z, y\rangle + \langle x, z\cdot y - y\cdot z\rangle \\
&= \langle z\cdot x, y\rangle + \langle x, z\cdot y\rangle - \langle x\cdot z, y\rangle - \langle x, y\cdot z\rangle \\
&= \langle x\cdot z, y\rangle + \langle z, x\cdot y\rangle - \langle x\cdot z, y\rangle - \langle x, y\cdot z\rangle \\
&= \langle z, x\cdot y\rangle - \langle x, y\cdot z\rangle = \langle L_x y - {}^tL_y x, z\rangle\ .
\end{aligned}$$

Hence it follows that

$$U(x, y) = \tfrac{1}{2}(L_x y - {}^tL_y x) = \tfrac{1}{2}(L_y x - {}^tL_x y)\ ,$$

so that

$$\nabla_x y = \tfrac{1}{2}(L_x y - L_y x) + \tfrac{1}{2}(L_y x - {}^tL_x y) = \tfrac{1}{2}(L_x - {}^tL_x)y\ .$$

Proposition 2. *Let S_x be the symmetric part of L_x,* i.e., *let $S_x = \frac{1}{2}(L_x + {}^tL_x)$. Then we have*

(i) $$S_x y = S_y x,$$

and the curvature tensor R and the sectional curvature k are given by

(ii) $$R(x, y) = -[S_x, S_y],$$

(iii) $$k(x, y) = \frac{\|S_x y\|^2 - \langle S_x x, S_y y\rangle}{\|x\|^2\,\|y\|^2 - \langle x, y\rangle^2}\ ,$$

where $\|x\| = \sqrt{\langle x, x\rangle}$.

Proof. (i) is equivalent to (5). In fact we have

$$\begin{aligned} 2\langle S_x y, z\rangle &= \langle (L_x + {}^tL_x)y, z\rangle = \langle x\cdot y, z\rangle + \langle y, x\cdot z\rangle \\ &= \langle y\cdot x, z\rangle + \langle x, y\cdot z\rangle = \langle (L_y + {}^tL_y)x, z\rangle = 2\langle S_y x, z\rangle\ . \end{aligned}$$

Since $R(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$, by Proposition 1, (2) and (6) we get

$$\begin{aligned} R(x, y) &= \tfrac{1}{4}[L_x - {}^tL_x, L_y - {}^tL_y] - \tfrac{1}{2}(L_{[x,y]} - {}^tL_{[x,y]}) \\ &= \tfrac{1}{4}\{[L_x, L_y] - [L_x, {}^tL_y] - [{}^tL_x, L_y] \\ &\qquad + [{}^tL_x, {}^tL_y] - 2[L_x, L_y] + 2{}^t[L_x, L_y]\} \\ &= -\tfrac{1}{4}[L_x + {}^tL_x, L_y + {}^tL_y] = -[S_x, S_y]\ . \end{aligned}$$

From (i), (ii) we obtain

$$\begin{aligned} \langle R(x, y)y, x\rangle &= \langle -[S_x, S_y]y, x\rangle = \langle -S_x S_y y + S_y S_x y, x\rangle \\ &= \langle S_x y, S_y x\rangle - \langle S_y y, S_x x\rangle = \|S_x y\|^2 - \langle S_x x, S_y y\rangle\ , \end{aligned}$$

which together with $k(x, y) = \dfrac{\langle R(x, y)y, x\rangle}{\|x\|^2\,\|y\|^2 - \langle x, y\rangle}$ gives (iii).

A clan V is said to be *elementary* if V satisfies the following conditions:

(E.1) $$V = \{u\} + P \qquad \text{(direct sum of vector spaces)}\ ,$$

(E.2) $$u\cdot u = u\ , \qquad u \neq 0\ ,$$

(E.3) $$u\cdot p = \tfrac{1}{2}p \quad \text{and} \quad p\cdot u = 0 \quad \text{for } p \in P\ ,$$

(E.4) $$p\cdot q = \Phi(p, q)u \qquad \text{for } p, q \in P\ ,$$

where Φ is a positive definite symmetric bilinear form on P.

The domain Ω corresponding to an elementary clan is the interior of a paraboloid (cf. [5], [6]):

$$\Omega = \{au + p\,;\, a - \tfrac{1}{2}\Phi(p, p) > -1 \text{ for } a \in \boldsymbol{R},\ p \in P\} .$$

To prove our theorem, therefore, it suffices to show

Theorem. *Let V be a clan. Then the following conditions are equivalent*:

(i) *The sectional curvature $k < 0$.*

(ii) *V is an elementary clan.*

Proof. We first prove that (i) implies (ii). Since V is a clan, there exists a nonzero element $u \in V$ such that (cf. [5])

$$u \cdot u = u , \tag{7}$$

$$V \cdot \{u\} \subset \{u\} , \tag{8}$$

and moreover putting $P = \{p \in V\,;\, p \cdot u = 0\}$ we have:

$$V = \{u\} + P \qquad \text{(orthogonal decomposition)}, \tag{9}$$

(10) L_u leaves P invariant, and the eigenvalues of L_u on $P = 0$ or $\frac{1}{2}$.

Let p be an element in P such that $L_u p = 0$. By (7), (8) and (9) we obtain

$$\langle S_u u, q\rangle = \tfrac{1}{2}\langle (L_u + {}^tL_u)u, q\rangle = \tfrac{1}{2}\langle u, q\rangle + \tfrac{1}{2}\langle u, u\cdot q\rangle = 0$$

for all $q \in P$, so that $S_u u \in \{u\}$. Put $S_u u = \lambda u$ ($\lambda \in \boldsymbol{R}$). Then it follows from Proposition 2(i) that

$$\langle S_u u, S_p p\rangle = \langle \lambda u, S_p p\rangle = \lambda\langle S_p u, p\rangle = \lambda\langle S_u p, p\rangle = \lambda\langle u\cdot p, p\rangle = 0 .$$

Therefore by Proposition 2 (iii) we have

$$k(u, p)(\|u\|^2 \|p\|^2 - \langle u, p\rangle^2) = \|S_u p\|^2 - \langle S_u u, S_p p\rangle = \|S_u p\|^2 \geq 0 .$$

Since $k < 0$, we have $p = 0$. Hence it follows from (10) that the eigenvalues of L_u on P are equal to $\frac{1}{2}$. By [5] this means that

$$p \cdot q = \Phi(p, q)u \qquad \text{for } p, q \in P , \tag{11}$$

where Φ is a positive definite symmetric bilinear form on P. Since $\langle x, u\rangle = \alpha_0(x)$ for all $x \in V$, u is the principal idempotent of V and $V = \{u\} + P$ is the principal decomposition of V, [6]. Therefore V is an elementary clan.

Conversely we shall prove that (i) follows from (ii). Let $u_0 = \dfrac{1}{\sqrt{\alpha_0(u)}}u$, $p_1, \cdots, p_{n-1}$ be an orthonormal basis of V such that $p_i \in P$. Then we have

$$(12)\qquad \begin{aligned} u_0\cdot u_0 &= \frac{1}{\sqrt{\alpha_0(u)}}u_0\,, & p_i\cdot p_j &= \frac{\delta_{ij}}{\sqrt{\alpha_0(u)}}u_0\,,\\ u_0\cdot p_i &= \frac{1}{2\sqrt{\alpha_0(u)}}p_i\,, & p_i\cdot u_0 &= 0\,, \end{aligned}$$

δ_{ij} being Kronecker's delta. Let $x = \lambda_0 u_0 + \sum_{i=1}^{n-1}\lambda_i p_i$ and $y = \mu_0 u_0 + \sum_{i=1}^{n-1}\mu_i p_i$ be elements in V where $\lambda_j, \mu_j \in \boldsymbol{R}$. By (12) we get

$$(13)\qquad x\cdot y = \frac{\lambda_0\mu_0 + \sum_{i=1}^{n-1}\lambda_i\mu_i}{\sqrt{\alpha_0(u)}}u_0 + \sum_{i=1}^{n-1}\frac{\lambda_0\mu_i}{2\sqrt{\alpha_0(u)}}p_i\,,$$

and therefore

$$\begin{aligned} \langle S_x y, u_0\rangle &= \left\langle \frac{1}{2}(L_x + {}^tL_x)y, u_0\right\rangle = \frac{1}{2}\langle x\cdot y, u_0\rangle + \frac{1}{2}\langle y, x\cdot u_0\rangle\\ &= \frac{1}{2}\left\langle \frac{\lambda_0\mu_0 + \sum_{i=1}^{n-1}\lambda_i\mu_i}{\sqrt{\alpha_0(u)}}u_0 + \sum_{i=1}^{n-1}\frac{\lambda_0\mu_i}{2\sqrt{\alpha_0(u)}}p_i, u_0\right\rangle\\ &\quad + \frac{1}{2}\left\langle \mu_0 u_0 + \sum_{i=1}^{n-1}\mu_i p_i, \frac{\lambda_0}{\sqrt{\alpha_0(u)}}u_0\right\rangle\\ &= \frac{1}{2\sqrt{\alpha_0(u)}}\left(2\lambda_0\mu_0 + \sum_{i=1}^{n-1}\lambda_i\mu_i\right), \end{aligned}$$

$$\begin{aligned} \langle S_x y, p_k\rangle &= \left\langle \frac{1}{2}(L_x + {}^tL_x)y, p_k\right\rangle = \frac{1}{2}\langle x\cdot y, p_k\rangle + \frac{1}{2}\langle y, x\cdot p_k\rangle\\ &= \frac{1}{2}\left\langle \frac{\lambda_0\mu_0 + \sum_{i=1}^{n-1}\lambda_i\mu_i}{\sqrt{\alpha_0(u)}}u_0 + \sum_{i=1}^{n-1}\frac{\lambda_0\mu_i}{2\sqrt{\alpha_0(u)}}p_i, p_k\right\rangle\\ &\quad + \frac{1}{2}\left\langle \mu_0 u_0 + \sum_{i=1}^{n-1}\mu_i p_i, \frac{\lambda_k}{\sqrt{\alpha_0(u)}}u_0 + \frac{\lambda_0}{2\sqrt{\alpha_0(u)}}p_k\right\rangle\\ &= \frac{\lambda_0\mu_k + \mu_0\lambda_k}{2\sqrt{\alpha_0(u)}}\,. \end{aligned}$$

Thus

$$(14)\qquad S_x y = \frac{1}{2\sqrt{\alpha_0(u)}}\left\{\left(2\lambda_0\mu_0 + \sum_{i=1}^{n-1}\lambda_i\mu_i\right)u_0 + \sum_{i=1}^{n-1}(\lambda_0\mu_i + \mu_0\lambda_i)p_i\right\},$$

from which it follows that

$$\begin{aligned} &\|S_x y\|^2 - \langle S_x x, S_y y\rangle\\ &= \frac{1}{4\alpha_0(u)}\left\{\left(2\lambda_0\mu_0 + \sum_{i=1}^{n-1}\lambda_i\mu_i\right)^2 + \sum_{i=1}^{n-1}(\lambda_0\mu_i + \mu_0\lambda_i)^2\right. \end{aligned}$$

$$
\begin{aligned}
(15) \qquad & - \left(2\lambda_0^2 + \sum_{i=1}^{n-1} \lambda_i^2\right)\left(2\mu_0^2 + \sum_{i=1}^{n-1} \mu_i^2\right) - \sum_{i=1}^{n-1} 4\lambda_0\mu_0\lambda_i\mu_i \Big\} \\
& = -\frac{1}{4\alpha_0(u)} \left\{ \left(\sum_{i=1}^{n-1} \lambda_i^2\right)\left(\sum_{i=1}^{n-1} \mu_i^2\right) - \left(\sum_{i=1}^{n-1} \lambda_i\mu_i\right)^2 \right. \\
& \qquad \left. + \sum_{i=1}^{n-1} (\lambda_0\mu_i - \mu_0\lambda_i)^2 \right\} .
\end{aligned}
$$

Therefore, if x and y are linearly independent, then we have $k(x, y) < 0$ by Proposition 2 (iii) and Schwarz's inequality. Hence our theorem is completely proved.

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